Maths for Economics

9. Matrix Algebra

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1 Matrices and linear algebra

After solving many systems of linear equations, you will start to notice a pattern. The **coefficients** on each variable are the numbers you're focused on manipulating, trying to turn some of them into zeroes so that the variable disappears. So why not focus on the coefficients alone and ignore the variable names?

1.1 Matrix notation

A matrix is a collection of numbers arranged in a particular pattern that we can manipulate according to the same rules we manipulate equations. We write them in a rectangular grid surrounded by parentheses or brackets. These mean the same thing but I will use brackets:

$$\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 3 & 3 \end{bmatrix}$$

As you can see, matrices can have different **dimensions**. The dimensions of the matrices above are 2×2, 1×3, and 2×3 respectively. The first number counts the number of rows, and the second is the number of columns. A matrix with only 1 row is called a **row matrix** and a matrix with only 1 column is called a **column matrix**. Both of these things are also called **vectors**. The word vector doesn't usually specify anything more than a single dimension, whether it's written vertically or horizontally, but we call say **row vector** or **column vector** synonymously with row and column matrix to specify which direction the list of values should be written in.

We usually use capital letters to name matrices, and the same letter in lowercase with two subscripts to write an individual number within a matrix. The first subscript is the row and the second subscript is the column. So if we call the first matrix above A, then $a_{11} = 1$, $a_{12} = 4$. $a_{21} = 0$, and $a_{2,2} = 2$. Or, if we already have these values defined, we can define $A = [a_{ij}]$.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \left[a_{ij}\right] = \left(a_{ij}\right)$$

1.2 Matrix operations

We can do arithmetic with matrices in a somewhat similar way as we do with numbers:

- $[a_{ij}] + [b_{ij}] = [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}\ \forall\ i, j$.
- $cA = c[a_{ij}] = [ca_{ij}]$ for any constant $c \in \mathbb{R}$

Notice that when we add matrices together, they have to have the same size so that we can match up all corresponding pairs of numbers and add them together.

Matrix multiplication is more complicated:

$$\underset{m\times n}{\mathbf{A}}\ \underset{n\times p}{\mathbf{B}} = \underset{m\times p}{\mathbf{C}} \qquad c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$$

Two matrices can only be multiplied if the number of columns in the first is the same as the number of rows in the 2nd. That also means that we have to multiply matrices in a specific order: If we change the order, either we won't be able to multiply them at all or the result might change.

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

In this example, if we tried to multiply them in the other order, we wouldn't be able to at all. In the next example, changing the order changes the result:

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

Therefore, with matrix multiplication we know longer have the commutative property. We have to distinguish between **premultiplication** of A by C, which yields CA, and **postmultiplication**, which yields AC. We do still have the following properties though:

- Associativity: $(AB)C = A(BC)$
- Left distributive law: $A(B + C) = AB + AC$
- Right distributive law: $(B + C)A = BA + CA$

1.3 Identity matrix

It's obvious from the definition of matrix addition that adding a matrix full of zeroes will always leave another matrix unchanged, just like adding 0 to any real number will leave it unchanged. And when we multiply any real number by 1, we also leave it unchanged. But under matrix multiplication, what matrix can we always multiply by another and leave the target unchanged?

The **identity matrix** can be premultiplied or postmultiplied by any matrix A and the result will be A:

$$\mathbf{I}_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

We have to choose the right size identity matrix to use in order to be able to multiply it with another matrix, but if we do, it is always true that $A \cdot I = I \cdot A = A$.

1.4 Matrix transpose and inverse

There are two more common operations we do to matrices. Transposition is simply a type of reflection:

$$[a_{ij}]^\top = [a_{ji}]$$

For example,

- $\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}^\top = \begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix}$
- $\begin{bmatrix} 1 & 4 & 2 \end{bmatrix}^\top = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$
- $\begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 3 & 3 \end{bmatrix}^\top = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 3 \end{bmatrix}$

The other operation is the inverse. Just like we can multiply a real number x with $\frac{1}{x}$ in order to get 1 back, we can usually find a matrix A^{-1} to multiply by A and get the identity matrix back. A matrix A^{-1} is the inverse of A if

$$A^{-1}A = AA^{-1} = I.$$

In this example, $A^{-1} = B$ and $B^{-1} = B$.

$$\text{If } A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

$$\text{Then } AB = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\text{And } BA = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

There are a few things to know about matrix inverses:

- Just like with functions, the -1 superscript does *not* mean to take a reciprocal like we do with real numbers in, for example, $3^{-1} = \frac{1}{3}$. With matrices it's not even possible to take a reciprocal because we only know how to multiply matrices, not divide them.

- Only square matrices can have inverses. Notice that the definition of an inverse says that it must yield I when *either* premultiplied or postmultiplied by the original matrix. Since we can only switch the order of matrix multiplication if the matrices are square, non-square matrices can't possibly have inverses.

- Even square matrices might not have inverses. With real numbers, every number except 0 has an inverse. But for the number zero, it's not possible to multiply zero by anything else to get 1. With matrices, there are many matrices that behave like the number 0 and can't be multiplied by anything to get the identity. We'll learn how to tell whether a matrix has an inverse in the next section.

- Matrix inverses are hard to calculate by hand. For a 2×2 matrix we can use a simple formula:
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
But, for larger matrices, the formulas for inverses get very large and cumbersome. I recommend using mathematical software like mathics or octave (both free) to calculate inverses of larger matrices.

There are some shortcuts for calculations with determinants and inverses that are useful to know. If A and B have compatible dimensions for the following expressions to be calculable, then:

- $(A^{-1})^{-1}$ and $(A^{\top})^{\top}$
- $(AB)^{-1} = B^{-1}A^{-1}$ and $(AB)^{\top} = B^{\top}A^{\top}$
- $(A^{\top})^{-1} = (A^{-1})^{\top}$
- $(cA)^{-1} = c^{-1}A^{-1} = \frac{1}{c}A^{-1}$ where c is any non-zero real number.

1.5 Determinants

One last property of square matrices that is probably the most useful thing to know about a (square) matrix is its determinant. We write the determinant of a matrix A either with the "det" function, or with vertical lines, just like we use for absolute values of real numbers:

$$\det(A) = |A|.$$

This notation isn't a coincidence. The determinant tells us whether a square matrix has an inverse: If the determinant is 0, it does not have an inverse. Otherwise it does. Now remember the real numbers: The only real number without an inverse is 0. Every other real number has an absolute value that isn't 0, and has an inverse.

Determinants, like inverses, are complicated to calculate by hand. For 2×2 matrices we have a simple formula

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

but for larger matrices, the formulas get more cumbersome.

Notice the connection between this formula and the formula for an inverse of a 2×2 matrix from the previous section. If the determinant is 0, then we can't actually use the formula for the inverse because we would have to divide by 0. Otherwise, we can. And so at least for 2×2 matrices, we can see that a nonzero determinant is a necessary and sufficient condition for the inverse to exist.

There are a few rules for determinants that are useful to know. If A and B are square matrices with the same dimension, then:

- $|A| = 0$ if any row or column consists only of 0's.
- $|A| = |A^\top|$
- If any row or column is a multiple of another row or column, then $|A| = 0$
- $|AB| = |A||B|$
- $|cA| = c^n|A|$ where c is any real number and n is the dimension of A.

1.6 Matrix representations of systems of equations

Now we have finally learned enough about matrices to understand how we can use them to analyze systems of linear equations. Consider the following matrix equation:

$$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

When we multiply the left hand side, we get

$$\begin{bmatrix} x + 3y \\ 2x - y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

which is exactly equivalent to the system of equations:

$$\begin{cases} x + 3y = 5 \\ 2x - y = 3 \end{cases}$$

We can actually use this method to represent any system of linear equations. In general, if x_i is the i^{th} variable that we're trying to solve for, and $a_{ij} \in \mathbb{R}$ are the coefficients, we can rearrange any system of linear equations to be in this form:

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$\cdots\cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$$

and this can be rewritten in matrix form:

$$[a_{ij}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$
$$\Leftrightarrow AX = B$$

What can we do with this representation?

$$AX = B$$
$$\Rightarrow A^{-1}AX = A^{-1}B$$
$$\Rightarrow X = A^{-1}B$$

This means that if A has an inverse, we can solve for every variable simultaneously by calculating $A^{-1}B$. And since we know that A has an inverse if an only if it has a nonzero determinant, we can figure out whether a system of equations has a solution by calculating $|A|$.

How do we represent the following systems of equations in matrix form? Which have solutions? What are they?

$$\begin{cases} x + 2y = 6 \\ 3x + 6y = 18 \end{cases}$$

$$\begin{cases} x + 2y = 6 \\ 3x + 6y = 10 \end{cases}$$

$$\begin{cases} x + 2y = 6 \\ 3x + 4y = 18 \end{cases}$$

2 Exercises

1. Let

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \\ 1 & 11 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 4 & 3 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

. Calculate the following, when possible:

(a) $a_{1,2}$ and $a_{2,1}$

(b) $A + B$

(c) $B + C$

(d) $A \cdot C$

(e) $C \cdot A$

(f) I_3

(g) $I_3 \cdot A$

(h) $I_3 \cdot C$

(i) $A^\top$

(j) $A^\top \cdot C^\top$

2. Verify that

$$A^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 0 & -1 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \\ 1 & 2 & -\frac{2}{3} \end{bmatrix}$$

3. Let
$$A = \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}$$
and
$$B = \begin{bmatrix} 0 & 3 \\ 3 & 2 \end{bmatrix}.$$
Calculate, when possible:
   (a) $|A|$
   (b) $2A^2$
   (c) $|B|$
   (d) A^{-1}
   (e) $A^\top$
   (f) $(A^\top)^{-1}$
   (g) $(A^{-1})^\top$
   (h) $|AB|$
   (i) $|BA|$

4. For each of the following systems of equations, a) represent the system using matrices, b) determine whether or not the system has a solution, c) find the solution, if possible, and d) explain your solution or lack of solution graphically, if possible.
   (a)
$$\begin{cases} 2x - y = 3 \\ 3x + 3y = 12 \end{cases}$$
   (b)
$$\begin{cases} 2x - 3 = y \\ 5y - x = 12 \end{cases}$$
   (c)
$$\begin{cases} x - y = 3 \\ 3x = 9 + 3y \end{cases}$$
   (d)
$$\begin{cases} x - 3 + 2y = 0 \\ 2x = 6 \end{cases}$$

(e)

$$\begin{cases} x - 3 + 2y = 0 \\ 3x + 6y = 3 \end{cases}$$

(f)

$$\begin{cases} 2x + z = 1 \\ y - x = 4 \\ y = 1 \end{cases}$$

(Hint: refer to number 2.)

(g)

$$\begin{cases} x + y + z = 1 \\ y + 3x - z = 3 \end{cases}$$