Maths for Economics

8. Systems of Equations

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1 Systems of equations

We've learned how to solve simple equations, and how to use simple functions to analyze economic questions. The same techniques can be used for more complicated questions too.

1.1 Systems of linear equations

Linear equations are the simplest to work with and still very powerful for modeling economic processes. We can actually use several linear functions simultaneously to understand more complicated questions, and they are still simple enough to work with.

For example, suppose that the quantity of pencils that a company is willing to sell at price p is $500p - 50$ (as long as the quantity is positive). At the same time, the number of pencils that people are willing to buy at price p is $1000 - 200p$. How can we find out which price will exactly clear the market? How many pencils q will be sold at that price?

The supply and demand equations are both linear equations, and we need to find values of p and q that satisfy both of them simultaneously.

$$\begin{cases} q = 500p - 50 \\ q = 1000 - 200p \end{cases}$$

The most straightforward way to handle systems of equations like these is to try to solve for one variable at a time. In this case, the first equation gives us a solution for q in terms of p. We can then use this solution in the 2nd equation:

$$q = 500p - 50 = 1000 - 200p \Rightarrow 700p = 1050 \Rightarrow p = 1.5$$

Now that we know p we can use either equation to figure out q: $q = 500p - 50 = 700$. 700 pencils will be sold for \$1.50 each.

1.2 Linear combinations of equations

This method of solving one equation at a time for each variable in turn will always work with linear systems, but sometimes there are shortcuts. Say we want to solve the following system:

$$\begin{cases} x + 3y = 5 \\ -x - y = -2 \end{cases}$$

Recall that we can preserve an equation by doing the same thing to both sides. We can always add 2 to both sides of an equation, for example. But if we know that $x = 2$, we can also add x to one side and 2 to the other. In the example above, we can add $x + 3y$ to one side of one equation, and 5 to the other, and equality is preserved because we know those two expressions describe the same number.

Let's do that to the second equation. This yields $(-x - y) + (x + 3y) = -2 + 5 \Rightarrow 2y = 3 \Rightarrow y = 1.5$. See what happened there: by adding the two equations together, we could immediately turn a system of equations with two variables into a single linear equation with only one variable. Here's a slightly more complicated example that illustrates the general principle:

$$\begin{cases} 3x = 5 - y \\ 7x = 2y \end{cases}$$

$$\Rightarrow \begin{cases} \frac{7}{3} \cdot 3x = \frac{7}{3}(5 - y) \\ 7x = 2y \end{cases}$$

$$\Rightarrow \begin{cases} 7x = \frac{35}{3} - \frac{7}{3}y \\ 7x = 2y \end{cases}$$

$$\Rightarrow 0 = \frac{35}{3} - \frac{13}{3}y \Rightarrow y = \frac{35}{13}$$

1.3 Other systems of equations

Other systems of equations might have functions other than linear functions in them, but the same approach of isolating one variable at a time will work. Often some trial and error is required to simply things.

Systems of equations might also have more than two variables. One equation is enough to determine the value of one variable, two (non-redundant) equations can determine the values of two variables, and so on.

2 Exercises

1. Solve the following systems of equations by eliminating one variable at a time:

   (a) $\begin{cases} 5x + y = 9 \\ 10x - 7y = -18 \end{cases}$

   $\Rightarrow y = 9 - 5x$
   $\Rightarrow 10x - 7(9-5x) = -18 \Leftrightarrow 10x - 63 + 35x = -18 \Leftrightarrow 45x = 45$
   $\Leftrightarrow x = 1 \Rightarrow y = 9 - 5 \cdot 1 = 4$

   (b) $\begin{cases} 5x + 30 = -4y \\ 3x - 9y + 18 = 0 \end{cases}$

   $\Rightarrow 3x = 9y - 18 \Rightarrow x = 3y - 6$
   $\Rightarrow 5(3y-6) + 30 = -4y \Rightarrow 15y - 30 - 30 = -4y \Rightarrow 19y = 0 \Rightarrow y = 0$
   $\Rightarrow x = 3 \cdot 0 - 6 = -6$

2. Solve the following systems of equations by combining equations into simpler relationships:

   $(3,4)$ (a) $\begin{cases} 3x - y = 5 \\ 2y - 3x = -1 \end{cases}$

   $(3x-y) + (2y-3x) = 5 + (-1) \Leftrightarrow y = 4 \Rightarrow 3x - 4 = 5$
   $\Rightarrow 3x = 9 \Rightarrow x = 3$

   $(3,1)$ (b) $\begin{cases} x + 3y = 6 \\ -3x + 7y = -2 \end{cases}$

   $\Rightarrow 3x + 9y = 18$
   $\Rightarrow 3x + 9y - 3x + 7y = 18 - 2 \Rightarrow 16y = 15 \Rightarrow y = 1 \Rightarrow x + 3 \cdot 1 = 6$
   $\Rightarrow x = 3$

   $(2,1)$ (c) $\begin{cases} 2x + 5y = 9 \\ 3x - 7y = -1 \end{cases}$

   $\Rightarrow 6x + 15y = 27$
   $\Rightarrow 6x - 14y = -2$
   $0x + 29y = 29 \Rightarrow y = 1 \Rightarrow 2x + 5 \cdot 1 = 9 \Rightarrow 2x = 4 \Rightarrow x = 2$

3. Solve the following systems of equations with any method:

   $(2,2,3)$ (a) $\begin{cases} x + y + z = 7 \\ 2x + y - z = 3 \\ 3x - 2y = 2 \end{cases}$

   $\Rightarrow z = 2x + y - 3$
   $\Rightarrow x + y + 2x + y - 3 = 7 \Rightarrow 3x + 2y = 10$
   $\Rightarrow 6x = 12 \Rightarrow x = 2 \Rightarrow 3 \cdot 2 + 2y = 10$
   $\Rightarrow 2y = 4$
   $\Rightarrow y = 2$
   $\Rightarrow z = 2 \cdot 2 + 2 - 3 = 3$

   $(1,2,1)$ (b) $\begin{cases} x + y + z = 4 \\ 3x + y - z = 4 \\ 2x - 2y + 4z = 2 \end{cases}$

   $\Rightarrow z = 3x + y - 4$
   $\Rightarrow x + y + 3x + y - 4 = 4 \Rightarrow 4x + 2y = 8 \Rightarrow y = 4 - 2x$
   $\Rightarrow 2x - 2y + 4(3x + y - 4) = 2$
   $\Rightarrow 2x - 2(4 - 2x) + 4(3x + 4 - 2x - 4) = 2$
   $\Rightarrow 2x - 8 + 4x + 12x + 16 - 8x - 16 = 2$
   $\Rightarrow 10x = 10 \Rightarrow x = 1$
   $\Rightarrow y = 4 - 2 \cdot 1 = 2$
   $\Rightarrow z = 3 \cdot 1 + 2 - 4 = 1$

   $(0,1)$ (c) $\begin{cases} e^x = y \\ x + \log 2 = \log(y + 1) \end{cases}$

   $\Rightarrow x + \log 2 = \log(e^x + 1)$
   $\Rightarrow x = \log\left(\frac{e^x + 1}{2}\right)$
   $\Rightarrow e^x = \frac{e^x + 1}{2}$
   $\Rightarrow e^x = 1 \Rightarrow x = 0 \Rightarrow y = 1$