Maths for Economics

7. Finance

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27. August 2015

1 Finance

We now know enough to understand how to mathematically analyze many financial situations. Let's start with a simply bank account with a 3% annual interest rate.

1.1 Effective annual interest rates

Year	Beginning Balance	Interest Earned	Ending Balance
1	$1,000.00	$30.00	$1,030.00
2	$1,030.00	$30.90	$1,060.90
3	$1,060.90	$31.83	$1,092.73
4	$1,092.73	$32.78	$1,125.51
5	$1,125.51	$33.77	$1,159.27
6	$1,159.27	$34.78	$1,194.05
7	$1,194.05	$35.82	$1,229.87
8	$1,229.87	$36.90	$1,266.77
9	$1,266.77	$38.00	$1,304.77
10	$1,304.77	$39.14	$1,343.92

At the end of each year, the interest earned gets added to the principal, and is thus able to earn its own interest the following year. But what if earned interest were able to earn its own interest sooner than after a full year?

Let's say the 3% annual interest payment were divided between twelve monthly payments of 0.25% each. The first year the balance would look like this:

Month	Beginning Balance	Interest Earned	Ending Balance
1	$1,000.00	$2.50	$1,002.50
2	$1,002.50	$2.51	$1,005.01
3	$1,005.01	$2.51	$1,007.52
4	$1,007.52	$2.52	$1,010.04
5	$1,010.04	$2.53	$1,012.56
6	$1,012.56	$2.53	$1,015.09
7	$1,015.09	$2.54	$1,017.63
8	$1,017.63	$2.54	$1,020.18
9	$1,020.18	$2.55	$1,022.73
10	$1,022.73	$2.56	$1,025.28
11	$1,025.28	$2.56	$1,027.85
12	$1,027.85	$2.57	$1,030.42

After one year, instead of \$30 of interest, the account has earned \$30.42. The **effective annual rate** of a 3% annual rate with monthly compounding is thus 3.042%, not 3%.

We can understand this relationship with functions too: If we start with x dollars in an account with a yearly interest rate of r, annually compounded, then the interest earned is rx. Adding this to the existing balance yields a yearend balance of $x + rx = (1+r)x$. The following year, we start with $(1+r)x$ dollars, and end up with $(1+r)(1+r)x = (1+r)^2x$. After three years, we have $(1+r)^3x$. And so a pattern emerges:

$$\text{Balance after } n \text{ years} = (1+r)^n x$$

If we compound the interest monthly, then each month an interest payment of $r/12$ is calculated. After one month we have a balance of $(1+\frac{r}{12})x$, and after two months a balance of $(1+\frac{r}{12})^2x$, and after a year $(1+\frac{r}{12})^n x$.

Overall, the effective annual interest rate of an account with an interest rate of r compounded n times per year is $(1+\frac{r}{n})^n$.

1.2 Continuous compounding

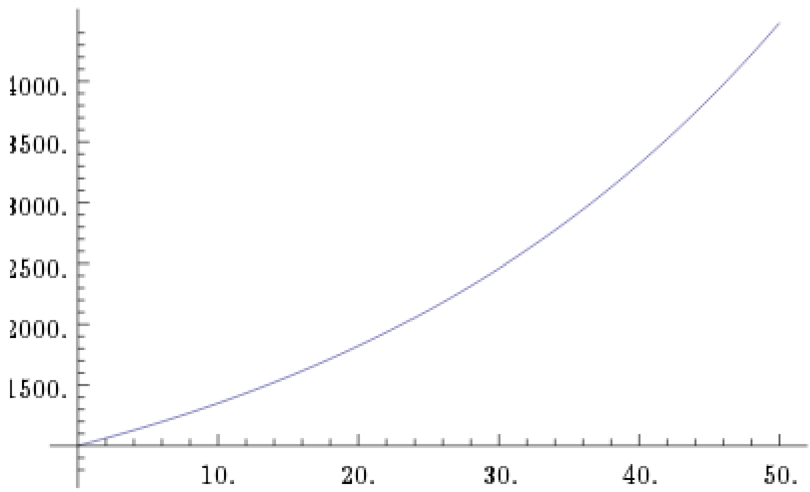
What happens if we increase the frequency of compounding even more?

If we compound interest daily, the balance at the end of the year will be $(1+\frac{.03}{365})^{365} \cdot 1000 = 1030.453$. If we compound every hour, it will be $(1+\frac{.03}{365 \cdot 24})^{365 \cdot 24} \cdot 1000 = 1030.454$. If we increase the frequency of compounding to **continuous compounding**, the number e finally shows up:

$$\text{Balance after } n \text{ years} = e^{rn} x$$

After one year, the balance in a continuously compounding 3% bank account with an initial deposit of \$1000 is $e^{.03 \cdot 1} \cdot 1000 = 1030.455$. This is where the value of $e = 2.71828182846...$ first shows up in economics.

The presence of an exponential function should tell you how powerful compounding interest is. Let's graph the contents of our 3% annual interest rate with continuous compounding for the next 50 years:



The rate of increase accelerates, and the account doubles its value in just over 20 years.

1.3 Present discounted value

What if we want to know how much we should be willing to pay for a bond that will pay \$1000 ten years from now. We have to compare the amount to our other option of putting our money in the bank. At a price p for the bond, we would be just willing to buy it if we expect that that money would grow to be worth \$1000 in ten years as well:

$$pe^{.03\cdot 10} = 1000 \Rightarrow x = 1000e^{-.03\cdot 10}.$$

In general, the **present discounted value** of an amount of money x paid in t years, when the continuously compounded interest rate is r, is

$$PDV(x) = xe^{-rt}.$$

Continuous compounding is the most common situation in the real world, but if our bank account instead compounds interest n times per year, the PDV formula would be

$$PDV(x) = x\left(\frac{1}{1+\frac{r}{n}}\right)^{nt}$$

which can be derived in the same way, by solving for the indifference price using the discretely-compounding interest functions from the previous subsection.

1.4 Annuities

A common situation in economics is to have a fixed amount of money be paid on a yearly (or other frequency) basis, continuing forever. A stock might pay a yearly dividend or a person might receive a fixed monthly social security payment, for example. But since each payment occurs at a different point in time, they have different PDVs. We have to assess the total value of the annuity by adding up the PDVs.

Consider an annual payment of \$1000, starting with the first payment made immediately, and imagine that we have access to a bank account that pays an effective annual interest rate of \$3%. Regardless of how frequently interest is compounded, this means that \$1000 deposited in the bank will be worth \$1030 in one year.

If this annuity is paid forever, it's currently worth

$$\begin{aligned} PDV(\$1000 \text{ annuity}) &= \textstyle\sum_{i=0}^{\infty} PDV(\$1000 \text{ in } i \text{ years}) \\ &= \textstyle\sum_{i=0}^{\infty} 1000 \left(\frac{1}{1+.03}\right)^{i} \\ &= 1000 \cdot \left(\frac{1}{1-\frac{1}{1.03}}\right) \\ &= 34,333 \end{aligned}$$

using the techniques for geometric series we learned about previously.

If the annuity isn't paid forever, we can use a similar method to figure out the current value of those payments. For example, suppose a homeowner needs to make 10 payments of \$1000 in order to finish paying their mortgage, one now and following payments yearly. How much does the bank value these payments right now?

$$\begin{aligned} PDV(10 \text{ payments of } \$1000) &= \textstyle\sum_{i=0}^{9} PDV(\$1000 \text{ in } i \text{ years}) \\ &= \textstyle\sum_{i=0}^{9} 1000 \left(\frac{1}{1+.03}\right)^{i} \\ &= 1000 \cdot \left(\frac{1-\left(\frac{1}{1.03}\right)^{10}}{1-\frac{1}{1.03}}\right) \\ &= 8,786 \end{aligned}$$

This same method is used to calculate the value of a stock that pays periodic dividends, or to calculate the value of a bond with a known payment schedule, or the value of almost any other financial instrument. The interest rate used is an interest rate that takes into account several factors such as inflation and the risk that the company will go bankrupt or default.

2 Exercises

1. What is the effective annual interest rate if the annual rate is 17% and interest payments are made
   (a) biannually
   (b) monthly
   (c) daily
   (d) continuously
2. What annual rate of growth is needed if

(a) A country with GDP \$500 million wants to reach \$5000 million within 10 years?

(b) A country wants to double its GDP within x years.

(c) A family puts \$100 into a continuously compounding savings account each year and wants to save \$2500 in 20 years.

3. Companies A, B, and C all sell a bond that will make yearly payments of \$10 in perpetuity. Company A is a company that has been around for hundreds of years without failing to make bond payments, so they are considered very low risk. Investors will allow them to borrow at a 1.5% interest rate. Company B can borrow at a 3% rate, and company C can borrow at a 6% rate. How much money can each company raise by selling these bonds?

4. With an interest rate of 3%, what is the PDV of each security?

   (a) Starting 10 years from now, annual payments of \$100 will be made for 20 years.

   (b) Starting 1 year from now, biannual payments of \$50 will be made for 30 years.

   (c) Starting immediately, 5 years of monthly payments of \$1000 will be made.

5. 794 million tons of iron were used in 1971. 249,000 million tons were available at that time. If consumption had increased by 5% per year, how many years would these reserves have lasted?