Maths for Economics

6. Manipulating Functions

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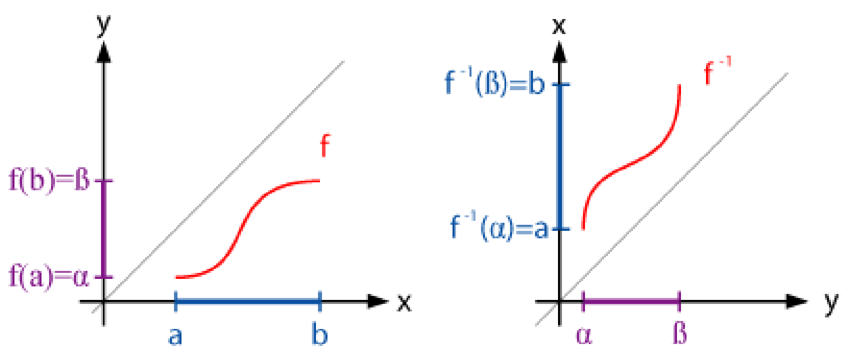
1 Using functions

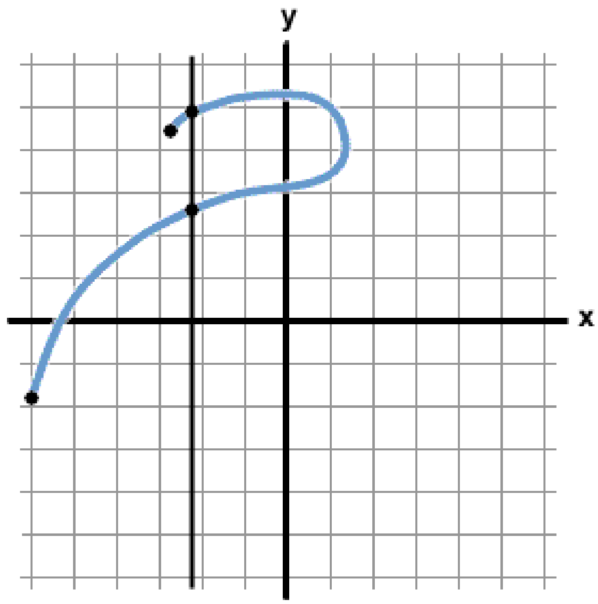
1.1 Inverse functions

The inverse of a function will take the *output* of a function and return the input that would generate that output. For example, if $f(5) = 3$ then $f^{-1}(3) = 5$.

It's important to note that while we use the -1 exponent to denote an inverse function, it is *not* usually true that $f^{-1}(x) = \frac{1}{f(x)}$. The exponent notation does not have its usual meaning when used with a function.



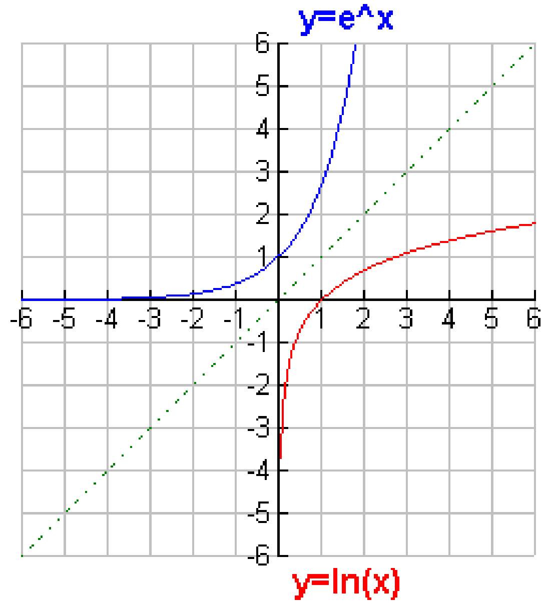
Recall that a function is a relationship between inputs and single outputs. Giving an input to a function will always produce a single output, but several different input values might lead to the same output. If we want to go the other direction from an output to inputs though, we won't be able to pick a unique answer if there are multiple input values that produce the same output.



This is why we have to be careful when we take square roots. Either -2 or 2 is a legitimate inverse of the point 4 under the function $f(x) = x^2$, but since functions can only have single output, the inverse function of $f(x) = x^2$ has to arbitrarily choose the positive value as $+2 = \sqrt{4}$.

Say a function f converts x values into y values; we can write $y = f(x)$. Then given a y value, we can find the value of $f'(y)$ by solving the function equation for x. For example, if $y = f(x) = x + 3$, then $x = f^{-1}(y) = y - 3$.

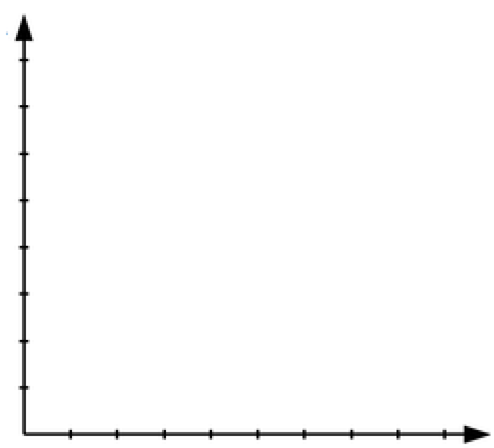
Visually, this looks like we are graphing the same function but with the horizontal and vertical axes switched. This is equivalent to reflecting a graph across the 45-degree line in the plane:



1.2 Functional inequalities

Functions are written with equations defining the relationships between variables. Sometimes we want to use functions to put bounds on quantities though. For example, if I have a certain budget I can spend, I don't have to buy goods adding up to exactly my budget, I just have to make sure that the total amount I'm spending is within my budget.

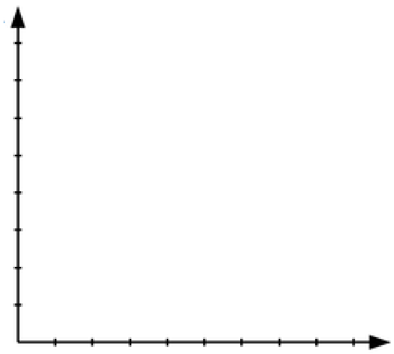
Just like we used inequalities to define subsets of the real line, and just like we use equalities to define subsets of the plane, we can use inequalities to define subsets of the plane. For example, a function $f(x) = x + 3$ defines a subset of the plane implicitly: $\{(x, y) \mid x \in \mathbb{R}, y = x + 3\}$. The inequality $y \leq x + 3$ defines a different subset of the plane: $\{(x, y) \mid x \in \mathbb{R}, y \leq x + 3\}$. These subsets are what we graph when we visualize inequality relationships.



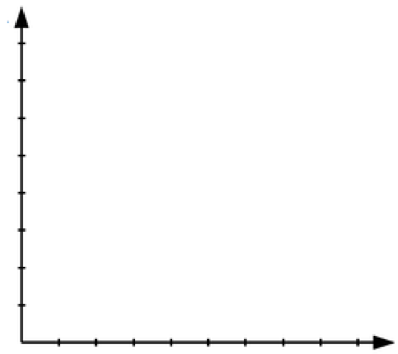
1.3 Function transformations

Functions can be built out of one another and modified in several ways. First, we can shift their graphs around in several ways:

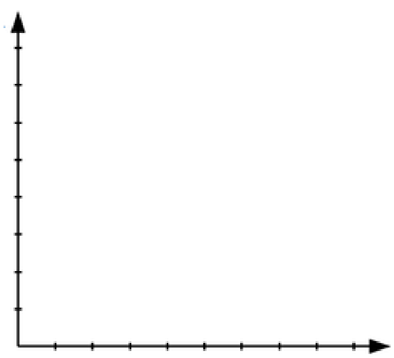
$f(x)+c$



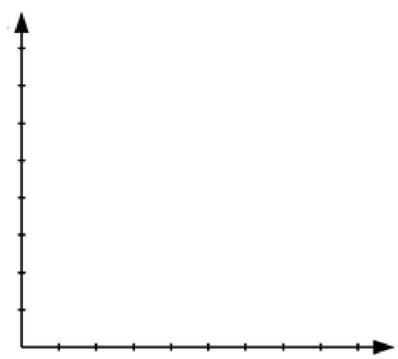
$cf(x)$



$f(x+c)$



$f(cx)$



We can also combine functions several operators:

- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(f \cdot g)(x) = f(x)g(x)$
- $(f/g)(x) = f(x)/g(x)$
- $(f \circ g)(x) = f(g(x))$

2 Exercises

1. Let $f(x) = x + 4$, $g(x) = x^2$, and $h(x) = -2x$. Find each of the following values:
   (a) $(f + g)(3)$
   (b) $(g \circ h)(1)$
   (c) $(h \circ g)(1)$
   (d) $(\frac{g+2}{3f})(0)$
   (e) $(g + fh)(2)$
2. Graph each of the following, using the same functions as in number 1:
   (a) $-2f(x)$ and $f(x)$
   (b) $g(x + 2)$ and $g(x)$
   (c) $h(x) - 2$ and $h(x)$
   (d) $g(x/3)$ and $g(x)$
3. Graph the inequalities:
   (a) $y > g(x)$
   (b) $y \leq g(x)$ and $y < f(x)$
4. Find and graph the following:
   (a) $(h \circ g)^{-1}(x)$
   (b) $h^{-1}(x)$
   (c) $f^{-1}(x)$ where $f(x) = \log(3x)$