Maths for Economics

4. Equations

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1 Equations

An equation is a statement that two different quantities represent the same point on the number line. In the simplest form, this might be something like $3 = 2 + 1$. Or, we might be told that $x = 1$. This tells us that even though x is an arbitrary letter, we know it represents the point 1 on the number line, so we can use it just like the number 1. Or, we might be told that $x + y = 1$. This gives us some information, but not enough to know exactly where on the number line x and y are pointing to. We only know that whatever numbers they represent, those numbers much sum to 1. It might be that $x = 0$ and $y = 1$, or that $x = -23,876$ and $y = 23,877$, or any one of infinitely many other possibilities.

1.1 Manipulating equations

If we know that two expressions describe the same number, we can manipulate those expressions in the same way, and they should still represent the same number.

- Adding or subtracting the same number from both sides of an equation will preserve the equality relationship. $3 = 2+1 \Rightarrow 3-1 = 2+1-1$, and $x = y+2 \Leftrightarrow x-1 = y+1$.
- Multiplying or dividing both sides of an equation by the same number will preserve the equality relationship. $3 = 2 + 1 \Rightarrow 6 = 2 \cdot (2 + 1)$, and $x = 2y \Rightarrow \frac{x}{2} = y$. *But!* it is very important to remember that dividing by zero is never allowed. I can conclude that $y = 1$ from $xy = x$ *only* if I'm sure that $x \neq 0$, and divide both sides of the equation by it knowing that I'm not dividing by zero.
- Raising each side of an equation to the same power will *usually* preserve the equality relationship. $3 = 2 + 1 \Rightarrow 3^2 = (2 + 1)^2$, and $x = 2y \Rightarrow x^2 = 4y^2$. *But!* there are important exceptions to this to remember. Since both 2^2 and $(-2)^2$ are 4, we can't take the square root of both sides of an equation. That is, $x^2 = y^2 \nRightarrow x = y$, because it might be that $x = 2$ and $y = -2$. Odd roots are ok, because $+1$ is the only third root of $+1$, but even roots have multiple possible answers in $\mathbb{R}$.

- It doesn't matter whether numbers or variables or letters are used. As always, everything fundamentally represents a point on the number line, and you can treat them all the same. You just have to make sure that what you're doing to a variable is valid for all of the possible numbers that variable might represent (like the dividing by zero example above.)

1.2 Solving equations

Usually, of course, we have a goal with our manipulations of equations. We usually want to figure out what number a particular variable represents. $x = 2$ tells us right away that x represents the number 2, but what about $\frac{x+2}{x-2} - \frac{8}{x^2-2x} = \frac{2}{x}$?

There are many ways to manipulate complicated equations like that into simple ones like $x = 2$. Here are some common, useful tricks.

1.2.1 Isolating the variable

You always want to isolate the variable on one side of the equation, and sometimes all you have to do that is to peel off the other numbers that it is interacting with in an expression. For example:

$$\begin{array}{rrcll}
 & 3x+7 & = & 13 & \text{starting equation} \\
\Rightarrow & 3x+7-7 & = & 13-7 & \text{subtract 7 from both sides} \\
\Rightarrow & 3x & = & 6 & \text{simplify} \\
\Rightarrow & \frac{3x}{3} & = & \frac{6}{3} & \text{divide both sides by 3} \\
\Rightarrow & x & = & 2 & \text{simplify}
\end{array}$$

1.2.2 Rational equations

Rational equations are equations with variables in the denominators of fractions. In these equations it's often helpful to get the variables entirely in the nominators first, by multiplying both sides by the denominators. For example:

$$\begin{array}{rrcll}
 & \frac{1}{2} + \frac{1}{x+2} & = & \frac{3}{4} & \text{starting equation} \\
\Rightarrow & (x+2)\cdot\left(\frac{1}{2} + \frac{1}{x+2}\right) & = & \frac{3}{4}(x+2) & \text{multiply both sides by } x+2 \\
\Rightarrow & (x+2)\cdot\frac{1}{2} + (x+2)\cdot\frac{1}{x+2} & = & \frac{3}{4}(x+2) & \text{use distributive property} \\
\Rightarrow & \frac{x+2}{2} + 1 & = & \frac{3}{4}(x+2) & \text{simplify} \\
\Rightarrow & 2\cdot\left(\frac{x+2}{2} + 1\right) & = & 2\cdot\frac{3}{4}(x+2) & \text{multiply both sides by 2} \\
\Rightarrow & 2\cdot\frac{x+2}{2} + 2 & = & \frac{3}{2}(x+2) & \text{use distributive property} \\
\Rightarrow & x+4 & = & \frac{3}{2}(x+2) & \text{simplify} \\
\Rightarrow & x+4 & = & \frac{3}{2}x+3 & \text{use distributive property} \\
\Rightarrow & 4 & = & \frac{x}{2}+3 & \text{subtract } x \text{ from both sides} \\
\Rightarrow & 1 & = & \frac{x}{2} & \text{subtract 3 from both sides} \\
\Rightarrow & 2 & = & x & \text{multiply both sides by 2}
\end{array}$$

1.2.3 Factoring

Suppose we have an equation $xy = 0$. The only way two numbers multiplied together can produce 0 is if one of them is 0, so we know that either $x = 0$, or $y = 0$, or both. We can often use this trick with more complicated equations:

- Suppose $xy = x$. By subtracting x from both sides and using the distributive property in reverse, we know that $xy - x = 0 \Rightarrow x(y - 1) = 0$. This tells us that either $x = 0$ or $y - 1 = 0$ (or both), so either $x = 0$ or $y = 1$ (or both).

- Suppose $x^2 = x$. We might be tempted to divide both sides of the equation by x, but remember, we can't divide by zero. We can check that x *might* be zero, because in that case the equation would be representing $0^2 = 0$, which is in fact true. But if it x isn't actually 0, we can proceed with dividing by x and make sure that our assumption that $x \neq 0$ isn't violated. This would give us $x = 1$.

  A different way to get these two possible values for x is to factor the equation. We can manipulate the equation as follows: $x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x - 1)$ which means that either $x = 0$ or $x - 1 = 0 \Rightarrow x = 1$, just as we hoped.

- Suppose $x^2 + 3x = -2$. We could try to factor the left hand side, but we can't use our trick with $x(x + 3) = -2$ because there are *many* ways two numbers can be multiplied together to get 3. But, there's another way we can factor the equation: $x^2 + 2x = -3 \Rightarrow x^2 + 2x + 3 = 0 \Rightarrow (x + 2)(x + 1) = 0$. We can use our trick with this equation to figure out that either $x + 2 = 0 \Rightarrow x = -2$, or that $x + 1 = 0 \Rightarrow x = -1$. Not every equation can be factored, and sometimes it takes some trial and error to figure out how to factor it, but when you can, this is an easy way to solve an equation with x^2 in it.

1.2.4 Completing the square

Suppose we have an equation $x^2 = 4$. We know there are only two values that you can square to get 4, so we know that $x = \pm\sqrt{4} = \pm 2$. We can use this approach to solve for x in more complicated equations too. We will often rely on the fact that $(x+b)^2 = x^2+2ax+a^2$; for example $(x+3)^2 = (x+3)(x+3) = x(x+3)+3(x+3) = x^2+3x+3x+9 = x^2+6x+9$. Therefore, if we recognize some terms in an equation as fitting into this pattern, we can factor them into a simpler $(x + a)^2$ term.

- Suppose $(x + 2)^2 = 9$. Then $x + 2 = \pm\sqrt{9} = \pm 3 \Rightarrow x = 1$.

- Suppose $x^2 + 2x = 8$. Here we can use the factorization trick mentioned above. We have an x^2 term and a $2x$ term, which can be the $2ax$ term above if $a = 1$. But if

$a = 1$ then we should have a $1^1 = 1$ term as well. We will have to create one:

$$\begin{aligned}
x^2 + 2x &= 8 && \text{original equation} \\
\Rightarrow \quad x^2 + 2x + 1 &= 9 && \text{add 1 to each side} \\
\Rightarrow \quad (x+1)^2 &= 9 && \text{factor left hand side} \\
\Rightarrow \quad (x+1) &= \pm 3 && \text{take square root of each side} \\
\Rightarrow \quad x &= 2 \text{ or } -4 && \text{subtract 1 from each side}
\end{aligned}$$

1.2.5 The quadratic formula

As you might guess, the technique used in the last example of completing the square can be used to solve *any* equation with nothing more complicated than an x^2 term! You can do it by completing the square, or you can use the quadratic formula, which is derived using the same method but might be easier to memorize and apply directly if you will use it frequently:

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sometimes this formula won't be usable because $b^2 - 4ac$ will be a negative number that we can't take the square root of. In these cases, there is simply no value of x on the real line that will satisfy the equation; that is, it's not an equation at all.[1] Also, note that if $a = 0$ we can't use the formula because we can't divide by 0. But in this case, there is no x^2 term in the equation as all and we can use simpler methods to solve for x.

- Example: If $3x^2 + x = 4$, then we can first subtract 4 from each side to yield $3x^2 + x - 4 = 0$, and then apply the quadratic formula: $x = \frac{-1 \pm \sqrt{1-4(3)(-4)}}{2*3} = \frac{-1 \pm \sqrt{49}}{6} = \frac{-1 \pm 7}{6} = -\frac{4}{3}$ or 1.

2 Exercises

1. Solve the following using factoring (if possible), completing the square, and the quadratic equation:
   (a) $x^2 + 2x = 3$
   (b) $2x^2 + 5x + 2 == 0$
   (c) $x^2 + 2x = 5$

[1] The solution actually does exist in a larger class of numbers called the complex numbers, but we won't study those.

2. Solve the following equations for x:

   (a) $2.5x + 0.45 = 0.95$
   (b) $6x - y = 2z$
   (c) $\sqrt{x+2} + 3 = 7$
   (d) $\frac{x}{x-5} = \frac{x+4}{x-6}$
   (e) $3x + 10 = x + 4$
   (f) $\frac{x+2}{x-2} - \frac{8}{x^2-2x} = \frac{2}{x}$
   (g) $\frac{1}{ax} + \frac{1}{bx} = 2$