Maths for Economics

3. Sequences

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1 Sums and series

One last bit of mathematical notation that will be very useful to economics is a way to shortcut writing long (or infinite) sums and products.

1.1 Sequences

First we need to know what a sequence is and how to write it. A sequence is just a list of numbers. For example, the sequence $1, 1, 1, 1...$ is a list of the number 1, repeated. A set can't specify how many times its members should be repeated, or in what order, so we need a different method to write sequences.

We can write these sequences in a similar way to sets though. We add a subscript to tell us which place in line each element takes:

$$1, 1, 1, 1... = \{a_i \mid a_i = 1\}$$

or

$$1, 3, 5, 7... = \{a_i \mid a_i = 2i - 1\}$$

We can also specify the length of the sequence by using subscripts and superscripts at the end to say which integer values i is allowed to be. If omitted, as above, we always assume that i can be any natural number. And because this i range is always either stated or implied, we can even take a shortcut of not writing a_i at all:

$$\{i + 1\}_0^5 = 1, 2, 3, 4, 5, 6$$

$$\{2k - 1\}_5^7 = 9, 11, 13$$

In this last example, I switched from using i to k, but since it's the only variable in the expression, we can infer that that is the letter that should take on each integer value between 5 and 7 inclusive.

1.2 Summation notation

Writing down the sum of a long sequence can be cumbersome. Luckily we can use summation notation as a shortcut:

$$\sum_{i=k}^{n} a_i = a_k + a_{k+1} + ... + a_{n-1} + a_n$$

Underneath the capital epsilon, we first specify which letter is the iteration variable. We also say what value it should start with. Above the summation symbol we say which number it should stop at.

Sometimes parts of this notation are omitted. For example, here it is implied that j should take on every natural number value:

$$\sum_{j} 2a_j = 2a_1 + 2a_2 + 2a_3 + ...$$

And in this statement we leave out the iteration variable since i is the only possibility:

$$\sum_{3}^{5}(i+1) = (3+1) + (4+1) + (5+1) = 15$$

It's always possible and unambiguous to write a summation with all parts included, and I encourage you to do so. But sometimes you will read summations that omit some parts, so in those cases you should infer what is meant according to these patterns.

Summations are just simple additions of numbers, so we can derive some rules for how they work. Try to convince yourself that these properties are true:

- Addition property: $\sum_{i=k}^{n}(a_i + b_i) = (\sum_{i=k}^{n} a_i) + (\sum_{i=k}^{n} b_i)$
- Homogeneity property: $\sum_{i=k}^{n} ca_i = c\sum_{i=k}^{n} a_i$
- $\sum_{i=k}^{n} c = (n-k+1)c$

And a very useful formula:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

1.3 Multiple sums

We can also nest summations. For example, we can calculate

$$
\begin{aligned}
& \textstyle\sum_{i=1}^{3}\sum_{j=1}^{4}(i+2j) \\
= \; & \textstyle\sum_{i=1}^{3}((i+2\cdot 1)+(i+2\cdot 2)+(i+2\cdot 3)+(i+2\cdot 4)) \\
= \; & \textstyle\sum_{i=1}^{3}(4i+20) \\
= \; & (4\cdot 1+20)+(4\cdot 2+20)+(4\cdot 3+20) \\
= \; & 84
\end{aligned}
$$

Note that we could also have used the addition property to split things up along the way:

$$
\begin{aligned}
& \textstyle\sum_{i=1}^{3}\sum_{j=1}^{4}(i+2j) \\
= \; & \textstyle\sum_{i=1}^{3}\left(\sum_{j=1}^{4} i+\sum_{j=1}^{4} 2j\right) \\
= \; & \textstyle\sum_{i=1}^{3}\left(4i+2\cdot\sum_{j=1}^{4} j\right) \\
= \; & \textstyle\sum_{i=1}^{3}(4i+20) \\
= \; & \textstyle\sum_{i=1}^{3} 4i+\sum_{i=1}^{3} 20 \\
= \; & \textstyle 4\sum_{i=1}^{3} i+3\cdot 20 \\
= \; & 4(1+2+3)+60 \\
= \; & 84
\end{aligned}
$$

(This method uses more steps when you write out each step, but is often easier to calculate in your head without writing everything down.)

In finite sums (that add together a finite sequence), you can switch the order of summation:

$$\sum_{i=k}^{n}\sum_{j=l}^{m} a_{ij}=\sum_{j=l}^{m}\sum_{i=k}^{n} a_{ij}$$

This doesn't always work when adding together infinite sequences! There are some really interesting reasons why, but they aren't so relevant to economics, so we won't study them in this course. Just remember to be careful when adding together infinitely many things.

1.4 Products of sequences

This notation is used less often in economics, but in case you see it somewhere, this is what it means:

$$\prod_{i=k}^{n} a_i = a_k \cdot a_{k+1} \cdot \ldots \cdot a_{n-1} \cdot a_n$$

1.5 Geometric Series

A geometric series is a sum of an infinite sequence of numbers that are related to each other by multiplication by some number. Of course, adding together infinitely many numbers often goes to ∞, but not always. It turns out that as long as each number in sequence is a multiple of the previous number by a factor with less than 1 absolute value, the sum is finite:

$$\sum_{i=0}^{\infty} k^i = \frac{1}{1-k}, \ |k| < 1$$

The most famous example of this equation is the answer to Zeno's paradox: When walking from point A to point B, we first have to walk half of the distance. Then we have to walk half of the remaining distance. Therefore we first have to walk $\frac{1}{2}$, then $\frac{1}{4}$, then $\frac{1}{8}$, then $\frac{1}{16}$... but if we keep doing this forever, we will make it there: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i - 1 = \frac{1}{1-\frac{1}{2}} - 1 = 2 - 1 = 1$.

Where does this equation come from? Notice that $(1-x)(1+x) = 1-x^2$. Similarly, $(1-x)(1+x+x^2) = 1-x^3$, and $(1-x)(1+x+x^2+x^3) = 1-x^4$. This is a general pattern: $(1-x)\sum_0^n x^n = 1-x^{n+1}$.

What happens if $x < 1$ and if we keep adding terms into this multiplication? x^{n+1} gets smaller and smaller, approaching zero. So we have $(1-x)\sum_0^{\infty} x^n = 1$. Rearranging this equation gives us the relationship we wanted to find.

Even if we only want to add together a small number of terms of this form, we can use this formula to make it easier. For example, if $|x| < 1$,

$$\begin{aligned} \textstyle\sum_{i=0}^{5} x^i &= (\textstyle\sum_{i=0}^{\infty} x^i) - (\textstyle\sum_{i=6}^{\infty} x^i) \\ &= \tfrac{1}{1-x} - x^6 (\textstyle\sum_{i=0}^{\infty} x^i) \\ &= \tfrac{1}{1-x} - x^6 \tfrac{1}{1-x} \\ &= \tfrac{1-x^6}{1-x} \end{aligned}$$

The general formula is

$$\sum_{i=0}^{i=k} = \frac{1-x^{k+1}}{1-x}.$$

These equations are going to be very useful in economics and we'll see some examples of why when we apply them to financial topics.

2 Exercises

1. Write down the following sequences with more precise sequence notation:

   (a) 3, 5, 7, 9, 11, ... $\{2i+1\}_{i=1}^{\infty}$

   (b) 100, 93, 86, ..., 58, 51 $\{100-7i\}_{i=0}^{7}$

   (c) 9, 27, 81, ... 2187, 6561 $\{3^i\}_{i=2}^{7}$

   (d) 8, 4, 2, 1, 0.5, ... $\{8\cdot(\frac{1}{2})^i\}_{i=0}^{\infty}$

2. Write the following sums with sum notation and compute the result:

   (a) The first 6 elements of sum **a** from number 1 above. $\sum_{i=1}^{6}(2i+1) = \left(2\sum_{i=1}^{6} i\right) + 6 = 2\cdot\frac{6\cdot 7}{2} + 6 = 48$

   (b) The sum of all elements in sum **b** above. $\sum_{i=0}^{7}(100-7i) = \sum_{i=0}^{7} 100 - 7\sum_{i=0}^{7} i = 800 - 7\cdot\frac{7\cdot 8}{2} = 604$

   (c) The first 4 elements in sum **c** above. $\sum_{i=2}^{7} 3^i = 9+27+81+243 = 360$

   (d) The sum of all elements in sum **d** above. $\sum_{i=0}^{\infty} 8\left(\frac{1}{2}\right)^i = 8\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i = 8\cdot\frac{1}{1-\frac{1}{2}} = 16$

3. Calculate the following sums:

   (a) $\sum_{i=2}^{\infty}\left(\frac{1}{2}\right)^i$ $\quad \left(\frac{1}{2}\right)^2\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i = \frac{1}{4}\cdot\frac{1}{1-\frac{1}{2}} = \frac{1}{4}\cdot 2 = \frac{1}{2}$

   (b) $\sum_{i=0}^{\infty}\frac{2}{3^i} = 2\sum_{i=0}^{\infty}\frac{1}{3^i} = 2\sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^i = 2\cdot\frac{1}{1-\frac{1}{3}} = 2\cdot\frac{1}{2/3} = 2\cdot\frac{3}{2} = 3$

   (c) $\sum_{i=1}^{\infty}\left(\frac{2}{3}\right)^i = \frac{2}{3}\sum_{i=0}^{\infty}\left(\frac{2}{3}\right)^i = \frac{2}{3}\cdot\frac{1}{1-2/3} = \frac{2}{3}\cdot\frac{1}{1/3} = \frac{2}{3}\cdot 3 = 2$

   (d) $\sum_{n=1}^{\infty}\frac{(-1)^n 2^{n-1}}{3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n 2^n\cdot 2^{-1}}{3^n} = \frac{1}{2}\sum_{n=1}^{\infty}\left(-\frac{2}{3}\right)^n = \frac{1}{2}\left(\sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n - 1\right)$

   $= \frac{1}{2}\left(\frac{1}{1-(-\frac{2}{3})} - 1\right)$

   $= \frac{1}{2}\left(\frac{1}{5/3} - 1\right) = \frac{1}{2}\left(-\frac{2}{5}\right) = -\frac{1}{5}$