Maths for Economics

1. Arithmetic and numbers

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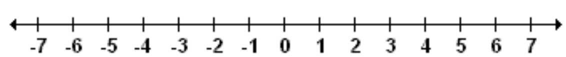
We're going to start off the course by reviewing some things you probably know and to learn about the symbols we use to talk about math. This won't be immediately applicable to economic problems, but before we can tackle those we need to get comfortable with the language, written and spoken, of math.

1 Numbers

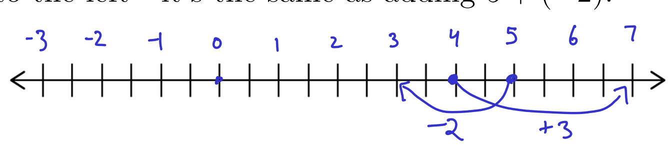
Math has many ways to describe numbers, types of numbers, and groups of numbers. But fundamentally, all of these ways refer to the same thing: points on a line.

1.1 The real number line

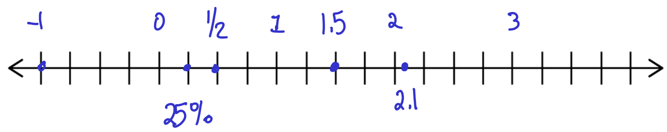


This is the number line. Every number is a point on this line, even if there isn't a label for it. Think of a meter stick: you can point to 5 micrometers, even if there isn't a tick mark for it. We can think about basic arithmetic operations in terms of the number line:

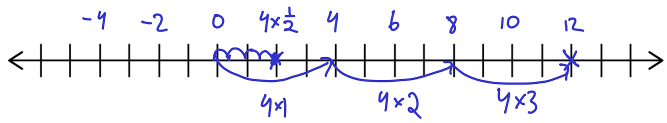
- Addition and subtraction happens by moving along the number line by the amount you want to add or subtract. $4 + 3$ asks where do you end up on the number line if you start at 4 and move to the right by 3? $5 - 2$ asks where you end up if you start at 5 and move 2 to the left - it's the same as adding $5 + (-2)$.



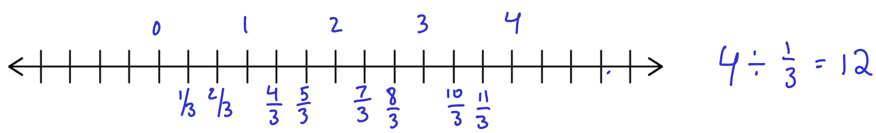
- Fractions split units on the number line into pieces; the fraction line simply represents division. Decimals also split units on the line into pieces. Percentages just move the decimal point over 2 places, which makes some numbers more convenient to talk about: 50%=0.5, 0.23=23%.



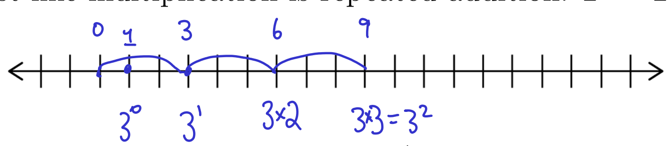
- Multiplication is repeated addition: $4 \times 3 = 4 + 4 + 4$. But it's more helpful to think of $\times$ as "of". 4×3 is 4 (groups) of 3, which is 12 total. And $4 \times \frac{1}{2}$ is 4 of $\frac{1}{2}$, which is 2. And $\frac{1}{2} \times \frac{1}{2}$ is $\frac{1}{2}$ of $\frac{1}{2}$, which is $\frac{1}{4}$. Note that multiplication is also sometimes written with a $\cdot$, or omitted completely. For example, $2 \cdot \sqrt{5} = 2\sqrt{5} = 2 \times \sqrt{5}$.



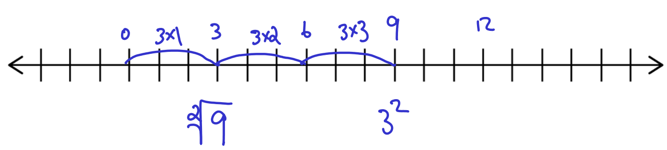
- Division is the opposite of multiplication, like subtraction is the opposite of addition: $12 \div 3$ asks "How many of 3 is 12?" and $4 \div \frac{1}{3}$ asks "How many of $\frac{1}{3}$ is 4?". But just like subtraction can be re-written as addition of a negative number, division can be re-written as multiplication by a fraction: $12 \div 3$ is the same as $12 \times \frac{1}{3}$, and $4 \div \frac{1}{3} = 4 \times \frac{1}{\frac{1}{3}} = 4 \times 3$. Division can also be written using fraction notation: $a \div b = \frac{a}{b}$.



- Exponents can also be illustrated on a number line. Exponentiation is repeated multiplication, just like multiplication is repeated addition: $2^3 = 2 \times 2 \times 2$.



- The opposite of exponentiation is taking roots.[1] And just like addition and subtraction, and like multiplication and division, we can write roots in terms of exponents too: $\sqrt[4]{16} = 16^{\frac{1}{4}}$ which asks what can you raise to the 4th power to get 16? Note that even thought both $(-2)^2$ and 2^2 are 4, $\sqrt{4}$ is said to be 2. If we want to specify either root, we can write $\pm\sqrt{4} = \pm 2$.

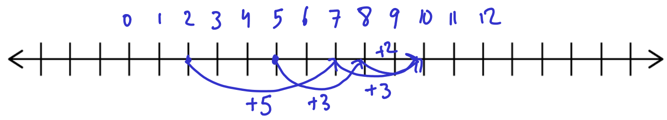


1.2 Rules of Arithmetic

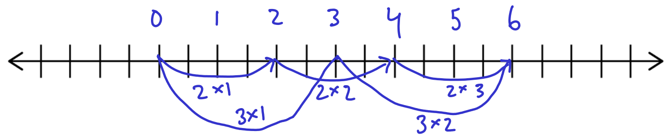
With the number line, we can make sense of the rules of arithmetic that you probably use every day.

- **Associativity** of addition and multiplication: Associativity just means that if you have to add or multiply 3 numbers, you can combine the first two first, or the second two, and you'll get the same answer. $2 + 5 + 3 = (2 + 5) + 3 = 2 + (5 + 3) = 10$, and $4 \times 3 \times \frac{2}{3} = (4 \times 3) \times \frac{2}{3} = 4 \times (3 \times \frac{2}{3}) = 8$.

[1] Almost: we'll see why this isn't quite true later in the course when we learn about inverse functions.



- **Commutativity** of addition and multiplication: Commutativity means that you can rearrange the order of things that you are adding or multiplying. $2+3=3+2=5$, and $2\times 3=3\times 2=6$.



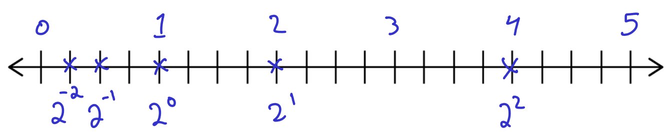
- Note that subtraction and division are not commutative! For that reason, it's useful to re-write them as addition and multiplication if you want to rearrange things. $3-2\neq 2-3$, but $3-2=3+(-2)=(-2)+3$, and while $4\div 2\neq 2\div 4$, $4\div 2=4\times\frac{1}{2}=\frac{1}{2}\times 4$.

- Also note that exponentiation does NOT have these properties! For example, $2^3=8\neq 9=3^2$, and $2^{(2^3)}=256\neq 64=(2^2)^3$.

- **Distributive** property: This property says that you can break up your multiplications into pieces. So $10\times 13=10\times(10+3)=10\times 10+10\times 3=100+30=130$. Or, $23\times 37=(20+3)\times(30+7)=20\times(30+7)+3\times(30+7)=20\times 30+20\times 7+3\times 30+3\times 7=600+140+90+21=851$. This is the method behind many mental speed math tricks.

- Powers of 0 and the 0^{th} power: 0 to any non-zero power is 0 multiplied by itself some number of times. This is clearly equal to 0.

  But, any non-zero number raised to the 0th power is a number multiplied by itself 0 times. What does that mean? If we think about where the different powers show up on the number line, you will see that x^0, for any non-zero value of x, will fit in with the other values if it's equal to 1.



  Note that since 0 to any power is 0, but any number to the 0th power is 1, we don't know how to define 0^0.

- Negative and fractional exponents: Following the same logic, we can extend exponentiation to negative exponents by *dividing* by a number a certain number of times: $x^{-n}=\frac{1}{x^n}$.

  By the definition of exponentiation, we can also derive the rules that $x^n x^m=x^{n+m}$ and that $(x^n)^m=x^{n\cdot m}$. Try to convince yourself that $x^{\frac{m}{n}}=\sqrt[n]{x^m}$. Also, $(x^m)^n=x^{(m\cdot n)}$

- The rules above can help us derive all kinds of other rules, but there are too many to memorize, so just always be sure to return to basics and think about the number line when doing complicated arithmetic. Remember though: $(a+b)^n\neq a^n+b^n$.

1.3 Order of operations

When doing calculations we also need to make sure to put the pieces together in the correct order. The correct order is:

1. *Parentheses*: First evaluate each parenthetical expression. Use the order of operations to evaluate each parenthetical expression as well.

2. *Exponents and roots*: Calculate all powers and roots. Roots come with implicit parentheses; evaluate these parenthetical expressions first. E.g. $\sqrt{2+3} = \sqrt{(2+3)}$.

3. *Multiplication and division*: Compute all multiplications and divisions, keeping the commutative property in mind. Note that fractions also come with implicit parentheses: $\frac{1}{2+3} = \frac{1}{(2+3)}$.

4. *Addition and subtraction*: Last of all, add together all numbers, keeping the commutative property in mind.

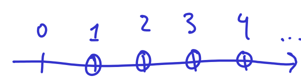
For example:

$$\begin{aligned}
&2 + \frac{3(1+2)^2}{9}\sqrt{8+(2^3)} \\
&= 2 + \frac{(3(1+2)^2)}{(9)}\sqrt{(8+(2^3))} \\
&= 2 + \frac{(3 \cdot 3^2)}{9}\sqrt{(8+8)} \\
&= 2 + \frac{(3 \cdot 9)}{9}\sqrt{16} \\
&= 2 + \frac{27}{9}\sqrt{16} \\
&= 2 + \frac{27}{9} \cdot 4 \\
&= 2 + 12 = 14
\end{aligned}$$

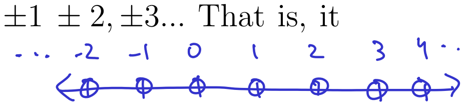
1.4 Types of numbers

All numbers are just parts of the same number line. But it's useful to give a few categories of numbers special symbols:

- $\mathbb{N}$: The **Natural** numbers. These are the ones we count with: 1, 2, 3...



- $\mathbb{Z}$: The **Integer** numbers. (Z stands for "zahlen" which is German for "number".) These are all the "whole" numbers on the number line: $0, \pm 1\ \pm 2, \pm 3...$ That is, it contains all of the natural numbers, their negatives, and 0.



- $\mathbb{Q}$: The **Rational** numbers. Q stands for quotient, because these are the numbers that we can write with fractions: $\frac{1}{2}$, 4, $3\,\frac{5}{73}$.

  Another piece of notation we can use to represent numbers on the numberline is the overline. This denotes a repeating decimal: $4.32\overline{14} = 4.32141414141414...$

  Rational numbers, when written in decimal form, always either terminate or repeat forever! (And, all numbers that do this are rational numbers.) For example, $0.\overline{9} = 1$, or $\frac{1}{3} = 0.\overline{3}$. Therefore, numbers that we write as decimals are also rational numbers, even if we don't know how to write them with a simple fraction.

- $\mathbb{R}$: The **Real** numbers. This is the full set of numbers on the number line. We can write them with decimal numbers, either terminating, or repeating, or going on forever without repeating, like $\pi = 3.141592653...$ or $\sqrt{2} = 1.41421356....$

1.5 Describing numbers

Here are some useful ways to talk about numbers:

- $<$: $x < y$ means that number x is *strictly less* than y. x lies to the left of y on the number line.
- $\leq$: $x \leq y$ means that either $x < y$ OR $x = y$. x is *weakly less* than y.
- $>$: $x > y$ means that $y < x$. x is *strictly greater* than y.
- $\geq$: $x \geq y$ means that $y \leq x$. x is *weakly greater* than y.
- $|x|$: The *absolute value* of x is the distance from x to 0 on the number line. $|-3| = |3| = 3$.

2 Exercises

1. Rewrite the following so that only natural numbers show up in exponents:

   (a) $\frac{a^{\frac{3}{8}}}{a^{\frac{1}{8}}}$ $\quad a^{3/8}\cdot\frac{1}{a^{1/8}} = a^{3/8}\cdot a^{-1/8} = a^{3/8-\frac{1}{8}} = a^{1/4} = \sqrt[4]{a}$

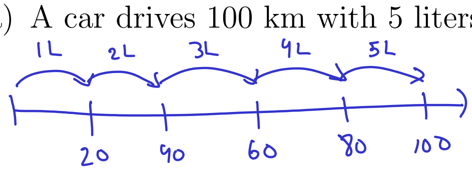
   (b) $(x^{\frac{1}{2}}x^{\frac{3}{2}}x^{-\frac{2}{3}})^{\frac{3}{4}}$ $\quad \left(\left(x^{1/2+3/2}\right)x^{-2/3}\right)^{3/4} = \left(x^{2}x^{-2/3}\right)^{3/4} = \left(x^{2-2/3}\right)^{3/4} = \left(x^{4/3}\right)^{3/4} = x^{\frac{4}{3}\cdot\frac{3}{4}} = x$

   (c) $\left(\frac{10p^{-1}q^{\frac{2}{3}}}{80p^{2}q^{-\frac{7}{3}}}\right)^{-\frac{2}{3}}$ $\quad \left(\frac{1}{8}\left(p^{-1}\cdot p^{-2}\right)\left(q^{2/3}q^{7/3}\right)\right)^{-2/3} = \left(2^{-3}p^{-3}q^{3}\right)^{-2/3} = \left(2^{-3}\right)^{-2/3}\left(p^{-3}\right)^{-2/3}\left(q^{3}\right)^{-2/3}$

   $= 2^{(-3)(-2/3)}p^{(-3)(-2/3)}q^{3(-2/3)} = 2^{2}p^{2}q^{-2} = \left(\frac{2p}{q}\right)^{2}$

2. Illustrate your answers to the following questions on a number line, and/or write an arithmetic expression that will give the answer:

   (a) A car drives 100 km with 5 liters of fuel. How many km/l did it get?



5 $\quad$ 100/5 = 20 km/l

mental math : $.35 \cdot 40 = \frac{35}{100} \cdot 40 = \frac{7 \cdot 5}{100} \cdot 40 = 7 \cdot \frac{1}{20} \cdot 40 = 7 \cdot 2$

(b) What is 35% of 40? $35\% \times 40 = .35 \cdot 40 = 14$

(c) A bank account starts with \$10 and doubles in size every 8 years. How much will be in it in 40 years? $10 \times 2 \times 2 \times 2 \times 2 \times 2 = 10 \cdot 2^5 = 320$

8 yrs 16 24 32 40

3. In each equation, decide whether the ? should be replaced with $=$ or $\neq$. You can assume that a and b are both positive integers.

(a) $\sqrt{25 \cdot 16}$? $\sqrt{25} \cdot \sqrt{16}$ $\quad =$ $\quad (ab)^n = a^n b^n$. $a = 25$, $b = 16$, $n = 1/2$

(b) $\sqrt{a+b}$? $\sqrt{a} + \sqrt{b}$ $\quad \neq$. Let $a = 1$, $b = 4$.

(c) $(a+b)^{\frac{1}{2}}$? $(\sqrt{a+b})^{-1}$ $\quad \neq$. $a^{1/n} = \sqrt[n]{a}$, not $\frac{1}{\sqrt[n]{a}}$.

4. Rewrite the following problems in a way that you could more easily calculate in your head (there are many possible answers):

(a) $237 - 40 + 3 = 237 + 3 + (-40) = 240 + (-40) = 200$

(b) $43 \times 8 = (40+3) \cdot 8 = 320 + 24 = 344$

(c) $29 \times 9 = 29 \cdot (10-1) = 290 - 29 = 290 - 30 + 1 = 261$