Maths for Economics

12. Calculus Concepts

Vera L. te Velde

1. September 2015

1 Concepts from Calculus

Calculus gives us a set of extremely powerful tools for analyzing general functions. Becoming entirely proficient with these tools requires extensive study and practice but a few concepts and definitions will at least enable you to read technical documents that use calculus and understand the logic of what they are doing.

2 Limits

Before anything else, we need to formalize the definition of a limit. We have already informally used limits, when we defined the number e: $(1+\frac{r}{n})^{nt}$ is the interest formula for a bank account compounding n times per year, and as n becomes extremely large, this actually equals e^{rt}. We can write this with limit notation:

$$\lim_{n\to\infty}(1+\frac{r}{n})^{nt} = e^{rt}$$

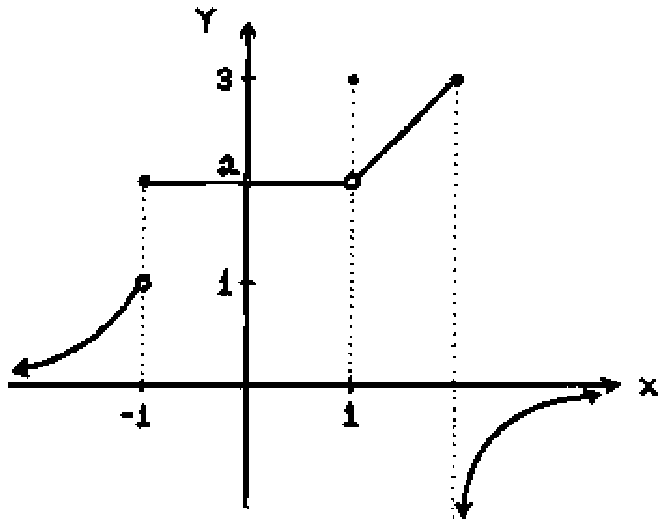
In general, limits are just what they sound like. The limit of a function as x approaches a is whatever the function gets closer and closer to as x gets close to (but not equal to) a.

Some limits are **one-sided**: If $f(x)$ approaches one value as x grows up to a, this is the left limit. Conversely, right limits require x to shrink down to a.

$$\text{Left limit}: \quad \lim_{x^-\to a} f(x) \qquad \text{Right limit}: \quad \lim_{x^+\to a} f(x)$$

Additionally, while $f(\infty)$ is not something we can evaluate, and while ∞ is not a value a function can produce, both of these are value in limit formulas as either the result or the target input value a.

These possibilities are illustrated in the following example:



In this diagram:

- $\lim_{x\to-\infty} f(x) = 0$
- $\lim_{x^-\to-1} f(x) = 1$
- $\lim_{x^+\to-1} f(x) = 2$
- $\lim_{x\to0} f(x) = 2$
- $\lim_{x\to1} f(x) = 2$
- $\lim_{x^-\to2} f(x) = 3$
- $\lim_{x^-\to2} f(x) = -\infty$
- $\lim_{x\to\infty} f(x) = 0$

Three more facts about limits:

- If a function is **continuous** at a point a, then $\lim_{x\to a} f(x) = f(x)$.
- Finite limits as x approaches $\pm\infty$ are called **horizontal asymptotes**.
- Limits as x approaches a finite value a that equal $\pm\infty$ are called **horizontal asymptotes**.

3 Derivatives

The next fundamental concept from calculus uses limits in its definition. A **derivative** of a function is a new function that outputs the slope of the original function. We write this in one of two main ways:

$$\text{derivative of } f(x) \text{ with respect to x :} \quad \frac{df}{dx}(x) = \frac{d}{dx}f(x) = f'(x)$$

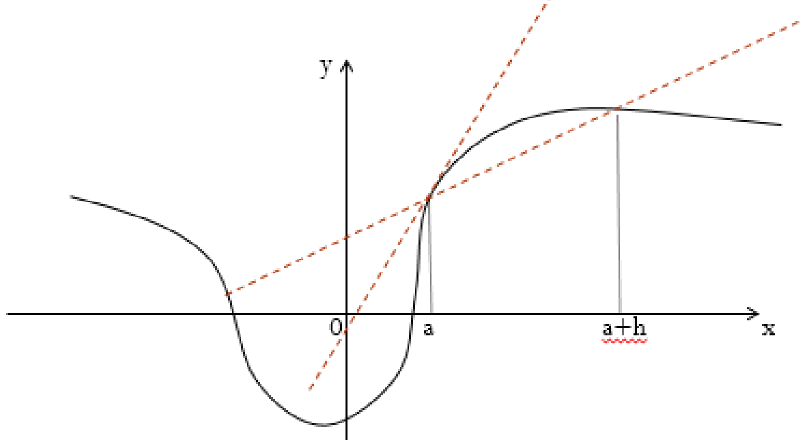
That is, the slope of $f(x)$ at the point $x = a$ is $f'(a)$.

For example, linear functions of the form $f(x) = ax + b$ have the same slope everywhere along the graph of the function, and its value is a. So, $f'(x) = a \; \forall \; x$ if f is a linear function.

Note that the slope is also frequently referred to as the **rate of change** of the function. Any time rates are discussed in practical applications, it's a good bet that that rate can be analyzed as a derivative of another function.

Not all functions have a slope that is as easy to calculate as it is for linear functions. For that reason, we have the formal definition of a derivative:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



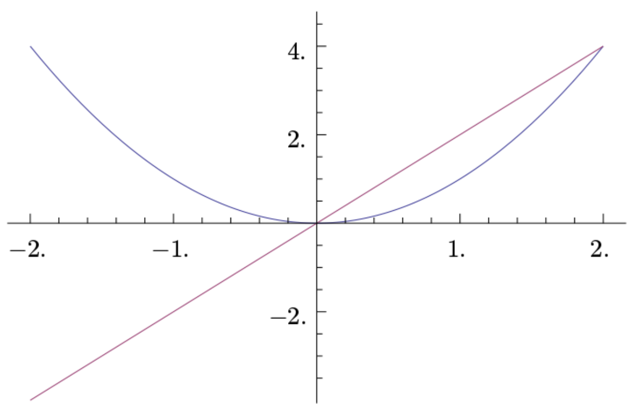
This definition is rarely used directly. Instead, the most common functions (including all the functions we have studied) have well-known derivative formulae. Here are the most important such formulae:

- $\frac{d}{dx}a = 0$
- $\frac{d}{dx}x^n = nx^{n-1}$ (for any $n \neq 0$)
- $\frac{d}{dx}e^x = e^x$
- $\frac{d}{dx}\ln x = \frac{1}{x}$

Even more powerfully, there are ways to combine derivatives, so that we can take derivatives of complicated functions by breaking them down into the pieces that we can use those formulae with. If f and g are functions and c is a constant, then:

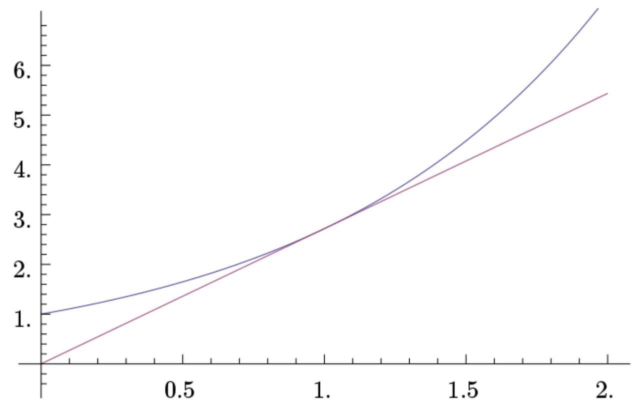
- $(f + g)'(x) = f'(x) + g'(x)$
- $(cf)'(x) = cf'(x)$
- Product rule: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
- Quotient rule: $(f/g)(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
- Chain rule: $(f \circ g)'(x) = f'(g(x))g'(x)$

We can check our derivative calculations by graphing both the original function and the derivative function, and checking that the derivative function is an accurate measure of the slope of the original function.



3.1 Approximation

One of the most useful things derivatives can be used for is approximating more complicated functions in order to be able to do analyze on the original function more easily. For example, GDP growth might be represented by an exponential function, but one a year-to-year basis it can easily be approximated with a linear function, which is much much easier to work with.



Linear approximations are quite straightforward. If we know the slope of f at a point $(a, f(a))$, that's all we need to know to determine the equation of a line that most closely fits f. The slope, we know by the definition of the derivative, is $f'(a)$. And using this in the linear function form at the point we know about, $f(a) = f'(a) \cdot a + b \Rightarrow b = f(a) - f'(a) \cdot a$. Therefore:

$$f(x \text{ at } a) \approx f'(a) \cdot x + f(a) - f'(a) \cdot a = f(a) + f'(a)(x - a)$$

If we want a more accurate approximate, we can correct this linear formula with an x^2 term, and then with an x^3 term, and so on. This process is called **Taylor series approximation.**

3.2 Optimization

Another of the most useful things we can use derivatives for is to analyze where functions achieve their maximum and minimum values.

Take the quadratic function $f(x) = x^2$ as an example. The minimum point on this function is identifiable because it's where the function turns around and starts going back up. What that means is that the slope goes from negative (downwards) to positive (upwards). The minimum point is right at the threshold, where the slope is exactly flat.

Recall that we called this turning point the *vertex* of a quadratic function, and we learned the formula for where this vertex occurs. Now we can see where this formula comes from:

$$\begin{gathered} \text{vertex of } f(x) = ax^2 + bx + c \\ = x \text{ s.t. } f'(x) = 0 \\ \Leftrightarrow 0 = \tfrac{d}{dx}(ax^2 + bx + c) \\ \Leftrightarrow 0 = 2ax + b \\ \Leftrightarrow x = -\tfrac{b}{2a} \end{gathered}$$

This same approach for finding turning points (called **local maxima** and **local minima**) of function works with any other function. Simply set the derivative equal to zero and verify that this is where the derivative changes signs, for example by comparing your answer to a graph.

4 Integration

Finally, **integration** is the opposite of differentiation. That is, integration and differentiation are inverse function. We write integrals as follows:

$$\int f(x)dx$$

Integration is harder to do by hand than differentiation, but there are some derivatives that are easy to undo. Referring back to our list of derivatives, we have:

- $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$ (for any $n \neq -1$)
- $\int e^x dx = e^x + C$
- $\int \frac{1}{x} dx = \ln x + C$
- $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

The $+C$ in each of these formulae reflects the fact that the derivative of *any* constant C is 0, so when we go backwards by integrating 0, we have to allow for the possibility that any value of C might have been part of the original function.

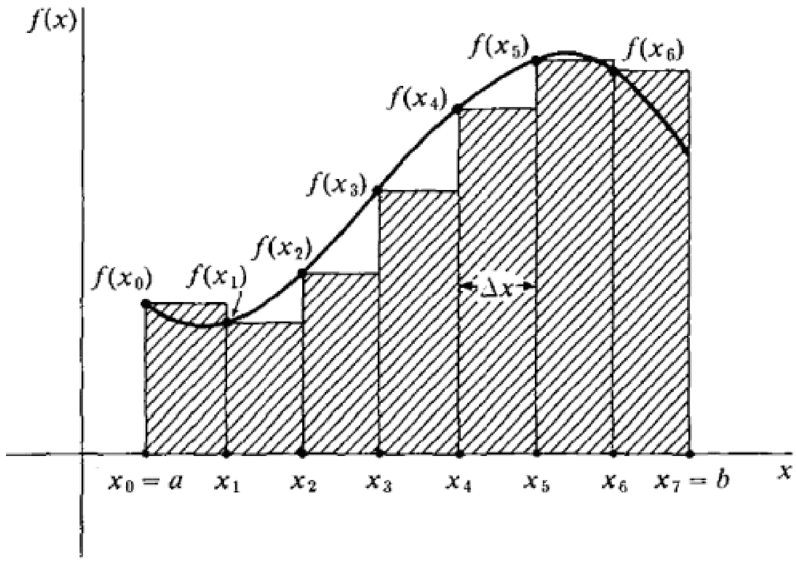
It turns out, what integration does is calculate areas beneath curves. The following notation specifies the area beneath a curve, between $x = a$ and $x = b$.

$$\int_a^b f(x)dx$$

This interpretation let's us define integrals with limits, like we did for derivatives.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{\frac{b-a}{n}} \frac{f(a + \frac{i}{n})}{n}$$

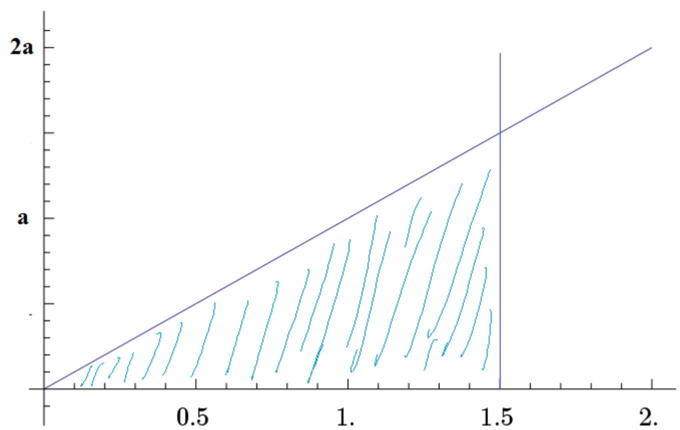
This is easier to understand with an illustration:



The way to evaluate these integrals with lower and upper limits is straightforward: first we calculate $F(x) = \int f(x)$, and then $\int_a^b f(x) = F(b) - F(a) = F(x)\Big|_a^b$. Notice the use of the vertical line with super and sub-script to represent the difference between a function at two points:

$$f(x)\Big|_a^b = f(b) - f(a)$$

For example, we can use integration to derive the formula for the area of a right triangle.



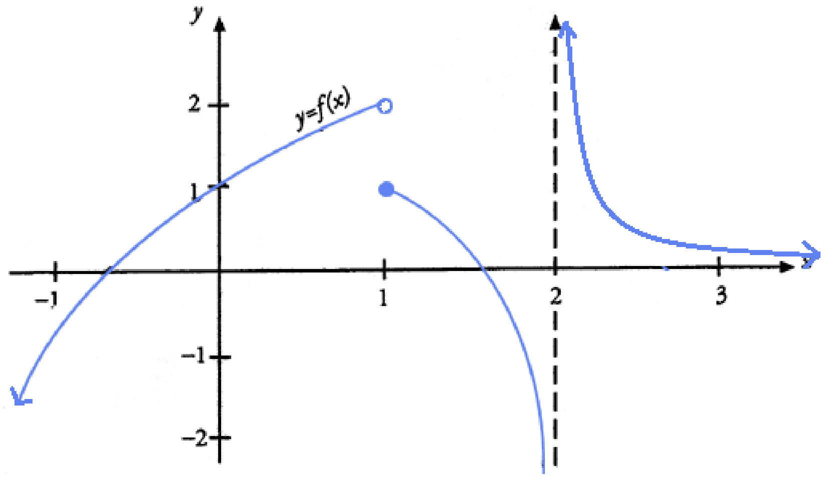
The area under the line $y = ax$, which goes through the point $(0, 0)$, is the area of a triangle. The height of the triangle is the height of the function ax, and the width of the triangle is x, the point at which we will stop integrating.

$$\int_0^x ax = \left(\frac{a}{2}x^2 + C\right)\Bigg|_0^x = (\frac{a}{2}x^2 + C) - (0 + C) = \frac{a}{2}x^2$$

This is equivalent to $(ax) \cdot (x)/2$, which is the base times height divided by 2 formula we're used to.

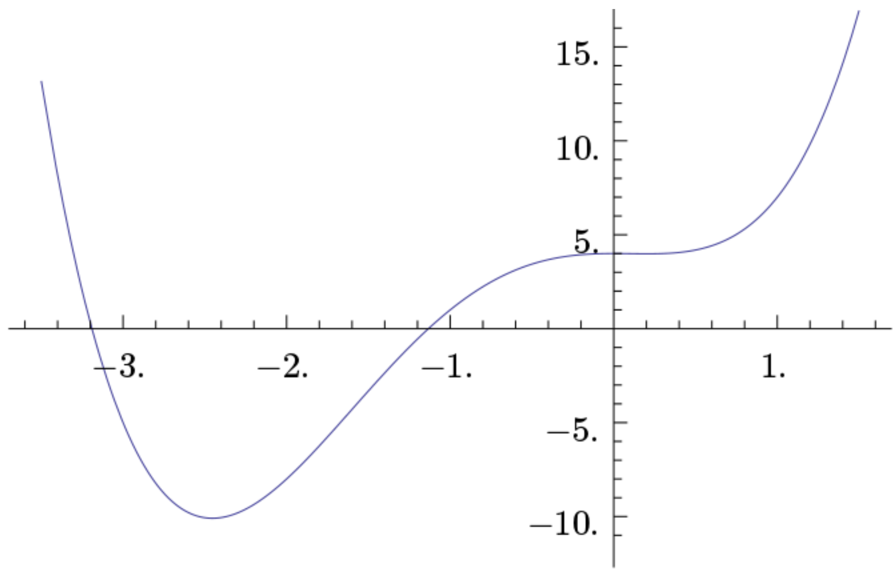
5 Exercises

1. What are the following limits? Use the graph below.



   (a) $\lim_{x \to -\infty} f(x)$
   (b) $\lim_{x \to 0} f(x)$
   (c) $\lim_{x^- \to 1} f(x)$
   (d) $\lim_{x^+ \to 1} f(x)$
   (e) $\lim_{x^- \to 2} f(x)$
   (f) $\lim_{x^+ \to 2} f(x)$
   (g) $\lim_{x \to \infty} f(x)$

2. Estimate the following derivatives and sketch the linear approximations of the function at each point, referring to the graph below.



(a) $f'(-3)$
(b) $f'(-2.5)$
(c) $f'(-1)$
(d) $f'(0)$

3. Calculate the following derivatives:

   (a) $f'(x)$ where $f(x) = x^2 - 3x$
   (b) $f'(0)$ where $f(x) = e^x + 1$
   (c) $f'(x)$ where $f(x) = xe^x$
   (d) $f'(1)$ where $f(x) = \log x^2$

4. What's the only point that might be a maximum or minimum value of the function $f(x) = \log x - x$?

5. Illustrate the area that is being calculated by the following integrals and calculate their values:

   (a) $\int_2^4 2dx$
   (b) $\int_{-1}^2 (x^2 + 1)dx$
   (c) $\int_0^1 e^x dx$